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Exact generation of epsilon-efficient solutions in multiple objective programming

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Abstract

It is a common characteristic of many multiple objective programming problems that the efficient solution set can only be identified in approximation: since this set often contains an infinite number of points, only a discrete representation can be computed, and due to numerical difficulties, each of these points itself might in general be only approximate to some efficient point. From among the various approximation concepts, this paper considers the notion of epsilon-efficiency which has also been shown to be of relevance other than merely for the purpose to approximate solutions. Following preceding work by the same authors, new generating methods are proposed to resolve various drawbacks of those methods derived earlier. Supporting theoretical results are established and the methods demonstrated on an engineering design example.

Keywords: multiple objective programming, epsilon-efficient solutions, epsilon-Pareto outcomes, approximation

1 Introduction

Mathematical optimization and decision making problems with multiple objectives have been studied for more than fifty years now, see Koopmans (1951), and many results have been established in theory, methodology and applications as collected in various monographs by

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Chankong and Haimes (1983), Sawaragi et al. (1985), Yu (1985), Steuer (1986) or more recently Miettinen (1999) and Ehrgott (2005). This paper joins the effort to promote the relevance of epsilon-efficient solutions introduced into multiple objective programming by Loridan (1984) and White (1986) and derives a methodology for their generation.

The general goal in any optimization or decision making process is to identify a single or all best solutions within a set of feasible points or alternatives. While it is theoretically possible to identify the complete set of best solutions, finding an exact description of this set often turns out to be practically impossible or at least computationally too expensive, and thus many research efforts focus on approximation concepts and procedures, see Ruzika and Wiecek (2005). Therefore, in practice the optimizing decision maker is frequently satisfied with suboptimal solutions, provided the loss in optimality can be justified by significant gain with respect to model simplicity and computational benefits. The consideration of such tradeoff reflects the common belief that the concept of epsilon-efficient solutions accounts for modeling limitations or computational inaccuracies, and thus is tolerable rather than desirable. Consequently, solution methods purposely avoiding efficiency while seeking to guarantee epsilon-efficiency have not been well developed.

Following earlier work by Engau and Wiecek (2005b) who investigate the significance of epsilon-efficient solutions in practical decision making situations, this paper continues the development of methods capable of generating epsilon-efficient solutions for multiple objective programming problems. Consider Figure 1.

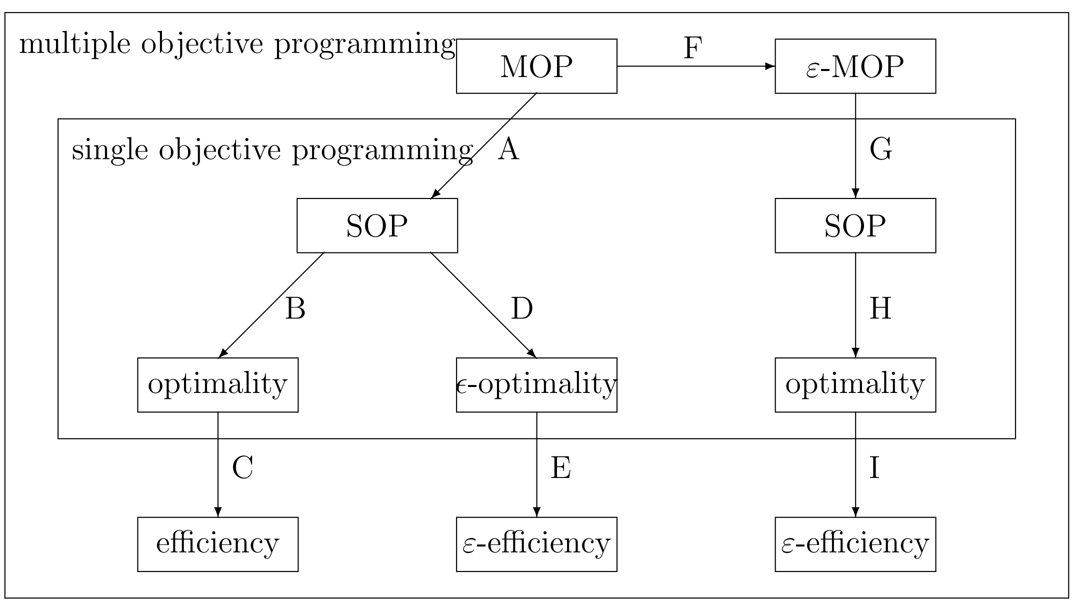

Figure 1: Generating methods for ε-efficient solutions

In the past, much effort has been undertaken to develop applicable methods to solve a multiple objective program MOP to efficiency, and perhaps the most prominent approach among those is to formulate an auxiliary single objective program SOP (A) which can be solved to optimality (B) using traditional linear or nonlinear optimization techniques and whose optimal solutions give rise to an associated efficient solution for the MOP (C). Based on the concepts of ϵ-optimality and ε-efficiency for single and multiple objective programming, respectively, several authors also investigated relationships between $\epsilon \in \mathbb{R}$ and $\varepsilon \in \mathbb{R}^m$ for which an ϵ-optimal solution for the SOP (D) produces an ε-efficient solution for the MOP (E), see Engau and Wiecek (2005b) and the references therein. However, these approximate methods suffer from the fact that an arbitrary ϵ-efficient solution for the SOP, in general, does not guarantee a full relaxation of efficiency for the MOP, while this may form the intended purpose of generating such solutions as mentioned above. In order to resolve this drawback, we propose alternative methods for the exact generation of those fully relaxed ε-efficient solutions for the MOP, and more precisely, we introduce four possible modifications of the original MOP into an ε-MOP (F) so that optimal solutions (H) for its associated scalarized problem (G) yield associated ε-efficient solutions (I) to the original MOP. Based on a thorough literature review, we believe that this approach and its relevance have not been studied before and thus are new to the research community.

The organization of the remaining text is now as follows. The adopted notation and basic terminology with special emphasis on the concepts of efficiency and ε-efficiency for the MOP and ϵ-optimality for its scalarized counterpart are established in Section 2 that also revisits and summarizes some prior results from Engau and Wiecek (2005b). Section 3 deals with the development of exact generating methods and gives further motivation and supporting theoretical results. In Section 4, these new methods are applied to an engineering design problem and compared to the previous approach in order to demonstrate the significant improvements achieved. Final remarks and possible future research directions are provided in the concluding Section 5.

2 Notation, basic terminology and prior results

We let $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, m$, be m real-valued objective functions and define the vector-valued function $f : \mathbb{R}^n \to \mathbb{R}^m$ by $f(x) := [f_1(x), \ldots, f_m(x)]^T$. Then we formulate the minimization problem

$$\text{MOP: minimize } f(x) \text{ subject to } x \in X \subseteq \mathbb{R}^n$$

as the multiple objective program of interest, for which X is a given set of feasible alternatives in the decision space $\mathbb{R}^n$. A feasible decision $x \in X$ is evaluated by the objective function f to produce the outcome $f(x)$ in the objective or outcome space $\mathbb{R}^m$. We define the set of all attainable outcomes as the image of X under f and denote this set by

$$Y := f(X) := \{y \in \mathbb{R}^m : y = f(x) \text{ for some } x \in X\}.$$

2.1 Efficient and ε-efficient solutions

We first define the underlying concept of minimality on $\mathbb{R}^m$. Given two outcomes y^1 and $y^2 \in Y \subseteq \mathbb{R}^m$, we write $y^1 \leqq y^2$ if and only if $y^1_i \leq y^2_i$ for all $i = 1, \ldots, m$, $y^1 \leq y^2$ if and only if $y^1 \leqq y^2$ and $y^1 \neq y^2$, $y^1 < y^2$ if and only if $y^1_i < y^2_i$ for all $i = 1, \ldots, m$, and $y^1 \geqq y^2$, $y^1 \geq y^2$, $y^1 > y^2$ if and only if $y^2 \leqq y^1$, $y^2 \leq y^1$, $y^2 < y^1$, respectively. Then $x^\circ \in X$ is called an efficient or weakly efficient decision for the MOP if and only if there does not exist another feasible $x \in X$ such that $f(x) \leq f(x^\circ)$ or $f(x) < f(x^\circ)$, and the sets of all efficient or weakly efficient solutions are denoted by X° or X°_w, respectively. An efficient or weakly efficient solution $x^\circ \in X^\circ$ or X°_w produces a Pareto or weak Pareto outcome $y^\circ = f(x^\circ)$, and the sets of all Pareto or weakly Pareto outcomes are denoted by Y° or Y°_w, respectively.

Solving the MOP now means to find the set (or a subset) of (weakly) efficient decisions or (weak) Pareto outcomes. Conditions for the existence of these solutions are established by Hartley (1978), Corley (1980), Borwein (1983) and Sawaragi et al. (1985) and, in general, can be guaranteed under certain compactness assumptions on the outcome set Y. For a comparison of several existence results, we refer to the recent survey provided by Sonntag and Zalinescu (2000).

Although theoretical studies have derived convenient representations for the set of Pareto outcomes in the objective space, it is important to realize that given a Pareto outcome $y^\circ \in Y^\circ$ or any other attainable outcome $y \in Y$, it is in general not possible to easily identify an efficient decision $x^\circ \in X^\circ$ or feasible decision $x \in X$ that produces this outcome $y^\circ = f(x^\circ)$ or $y = f(x)$, respectively.

Given $\varepsilon \in \mathbb{R}^m$, $\varepsilon \geqq 0$, a feasible decision $x^\circ \in X$ is called an ε-efficient or weakly ε-efficient solution for the MOP if and only if there does not exist another feasible $x \in X$ such that $f(x) \leq f(x^\circ) - \varepsilon$ or $f(x) < f(x^\circ) - \varepsilon$, and the sets of all ε-efficient or weakly ε-efficient solutions are denoted by $X^{(\varepsilon)}$ or $X_w^{(\varepsilon)}$, respectively. An ε-efficient or weakly ε-efficient solution $x^\circ \in X^{(\varepsilon)}$ or $X_w^{(\varepsilon)}$ produces an ε-Pareto or weak ε-Pareto outcome $y^\circ = f(x^\circ)$, and the sets of all ε-Pareto or weakly ε-Pareto outcomes are denoted by $Y^{(\varepsilon)}$ or $Y_w^{(\varepsilon)}$, respectively. In particular, we obtain that $X^{(0)} = X^\circ$, $X_w^{(0)} = X_w^\circ$, $Y^{(0)} = Y^\circ$ and $Y_w^{(0)} = Y_w^\circ$.

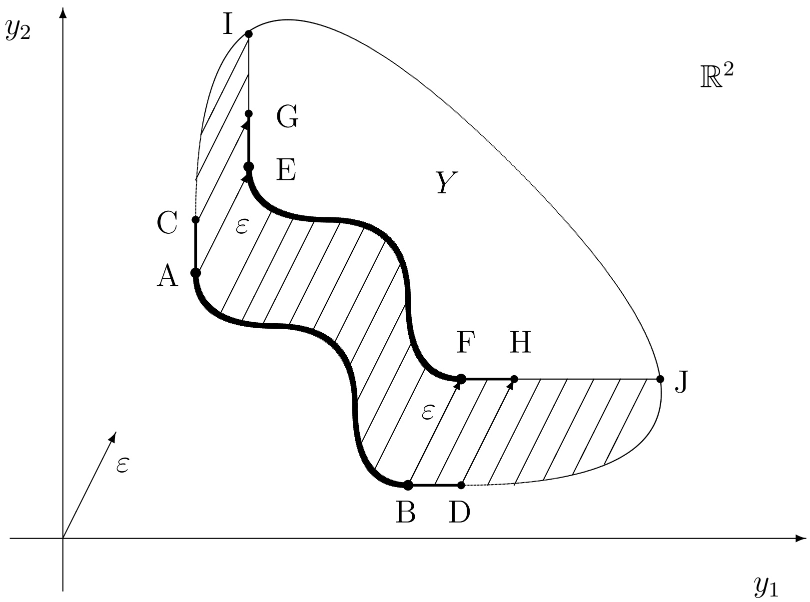

Figure 2: Sets of (weakly) Pareto and (weakly) ε-Pareto outcomes

Figure 2 illustrates the relationships between the different solutions concepts for a generic outcome set of some biobjective problem. Consider the depicted set $Y \subseteq \mathbb{R}^2$. The set Y° of Pareto outcomes is given by the curve connecting points A and B, while all weakly Pareto outcomes of the set Y_w° lie on the extended curve from C to D. Adding the given vector $\varepsilon \in \mathbb{R}^2$ now shifts the points A, B, C and D to E, F, G and H, respectively, and then the set $Y^{(\varepsilon)}$ of ε-Pareto outcomes is the shaded area enclosed by all marked points without the curve connecting I, G, E, F, H and J, while the set $Y_w^{(\varepsilon)}$ of weak ε-Pareto outcomes includes the curve from I to J. It follows that for this particular example, the set $Y_w^{(\varepsilon)}$ is closed, while the set $Y^{(\varepsilon)}$ of ε-Pareto outcomes is neither closed nor open. Consequences of this last observation are further addressed at the beginning of Section 3.

2.2 Scalarizing the MOP and ϵ-optimal solutions

The most frequent approach to solving MOPs is to use a parametric characterization of the set of efficient decisions by aggregating the objective functions f_i, $i = 1, \ldots, m$, into one real-valued objective s which is minimized if and only if the associated decision $x \in X$ is (weakly) efficient for the original MOP. We call such an objective function s a scalarization function for the MOP and require that this function preserves the partial order defined on $Y \subseteq \mathbb{R}^m$.

Let y^1 and $y^2 \in Y$ be any two attainable outcomes. Then we call the scalarization function s increasing if $y^1 \leqq y^2$ and $y^1 < y^2$ implies that $s(y^1) \leq s(y^2)$ and $s(y^1) < s(y^2)$, respectively. If, in addition, $y^1 \leq y^2$ implies that $s(y^1) < s(y^2)$, then s is called strictly increasing.

Based on several ideas and different methodological concepts, a large number of increasing scalarization functions is available, possibly depending on additional scalarization parameters or auxiliary variables. Common characteristics of these functions include the use of weighting coefficients as in the weighted sum approach by Geoffrion (1968), reference points as in Benson (1978) or in combination with norms such as the weighted and augmented ℓ_p norms as in Steuer (1986), objective function levels as in the constrained objective approach discussed in Chankong and Haimes (1983), search directions as in Pascoletti and Serafini (1984), aspiration levels as in goal programming approaches, or other combinations of the above. Wierzbicki (1986) examines several properties of scalarization functions, including their ability to generate the complete set of efficient decisions when varying the respective scalarization parameters. Li et al. (1999) derive relationships between these parameters for four different scalarization methods when generating the same efficient solution.

For the purpose of this paper, we formally define a scalarization function as an increasing function $s : Y \times \Pi \to \mathbb{R}$, where Π is the set of admissible scalarization parameters. Then the parametric single objective program associated with the MOP is the minimization problem

$$\text{minimize } s(y, \pi) \text{ subject to } y \in Y$$

with $\pi \in \Pi$ as fixed scalarization parameter. However, since the set Y is in general not given in explicit form, the minimization needs to be done over the feasible set X for the MOP, thus yielding the final formulation

$$\text{SOP}(\pi)\text{: minimize } s(f(x), \pi) \text{ subject to } x \in X.$$

Then $x^\circ \in X$ and $y^\circ = f(x^\circ) \in Y$ are called optimal solutions for the SOP if and only if $s(f(x^\circ), \pi) \leq s(f(x), \pi)$ for all $x \in X$. Given $\epsilon \in \mathbb{R}$, $\epsilon \geq 0$, then $x^\circ \in X$ and $y^\circ = f(x^\circ) \in Y$ are called ϵ-optimal or strictly ϵ-optimal solutions for the SOP if and only if $s(f(x^\circ), \pi) \leq$

$s(f(x), \pi) + \epsilon$ or $s(f(x^\circ), \pi) < s(f(x), \pi) + \epsilon$ for all $x \in X$, respectively.

Note that we define ε-efficient and weakly ε-efficient solutions for the MOP, but ϵ-optimal and strictly ϵ-optimal solutions for the SOP, in accordance with the standard terminology established in the literature, here namely Loridan (1982, 1984) and White (1986).

The following lemma illustrates the importance for the chosen scalarization function to be increasing in order to relate optimal solutions for the SOP and efficient decisions for the MOP and is similarly established by Wierzbicki (1986).

Lemma 2.1. *Let the* MOP *and an* SOP *with increasing scalarization function s be given. If $x^\circ \in X$ is optimal for the* SOP*, then x° is weakly efficient for the* MOP*. If s is strictly increasing, then x° is efficient.*

For different choices of the scalarization parameter $\pi \in \Pi$, in general we obtain different optimal solutions for the SOP and thus different efficient decisions for the MOP.

Two of the most popular scalarization methods are the weighted-sum and the weighted-Tchebycheff norm scalarization which we formulate now for later reference in the subsequent section.

Given the MOP, let $w \in \mathbb{R}^m, w \geq 0$ be a given vector of weights for the objective function $f = (f_1, \ldots, f_m)^T$. Then the weighted-sum scalarization for the MOP is defined as

$$\text{WS}(w)\text{: minimize } \sum_{i=1}^{m} w_i f_i(x) \text{ subject to } x \in X.$$

In addition to the above, let $r \in \mathbb{R}^m$ be a given reference point. Then the weighted-Tchebycheff norm scalarization for the MOP is defined as

$$\text{TN}(r, w)\text{: minimize } \max_{i=1,\ldots,m} \{w_i(f_i(x) - r_i)\} \text{ subject to } x \in X.$$

2.3 Approximate generating methods

As we mentioned before, it is theoretically possible to identify the complete set of efficient decisions. In practice, however, often only a discrete representation of the efficient set can actually be computed and, due to numerical difficulties, each of these points itself might only be approximate to some truly efficient point. In Engau and Wiecek (2005b), seven different scalarization methods are examined with respect to ϵ-optimality and ε-efficiency and corresponding relationships are established between $\epsilon \in \mathbb{R}$ and $\varepsilon \in \mathbb{R}^m$ when solving both the scalarized problem SOP for an ϵ-optimal and the original MOP for an ε-efficient solution.

Based on these results a generic procedure for the computation of ε-efficient decisions for the MOP is proposed, thereby relying on the idea that solving any of these SOPs with some scalarization parameter $\pi \in \Pi$ for a (strictly) ϵ-optimal solution also produces a (weakly) ε-efficient decision for the MOP, according to the path ADE in Figure 1.

This procedure assumes that based on prior knowledge and experience, a decision maker is able to choose both the vector parameter ε and a suitable SOP together with the required scalarization parameter π. Clearly, these choices depend on the actual type of problem to be solved and the desired relaxation of efficiency for the MOP. Furthermore, all results only imply an upper bound on the set of admissible values for ϵ such that an ϵ-optimal solution for the SOP is guaranteed to be (weakly) ε-efficient for the MOP, and hence, when solving the SOP for an ϵ-optimal solution, it is recommended that the actual ϵ should be chosen equal to the upper bound if the full relaxation is to be accomplished. However, while the decision maker may gain certain control over the ε-efficiency of generated feasible decisions for the MOP by varying the parameter ε, this approach highly depends on the capability to find reasonably relaxed ϵ-optimal solutions for the SOP. Although in principle accomplishable by a simple modification of the termination criterion of the optimization algorithms used, the lack of accurate bounds on the actual optimum and thus the uncertainty regarding the actual relaxation achieved motivates the development of methods for which ε-efficient decisions can be obtained as (exact) efficient decisions, thereby guaranteeing the full desired relaxation.

3 Exact generating methods

We formulate the particular problem of interest as to find the worst solutions among the set of ε-efficient decisions and then introduce four different approaches to modify the original MOP formulation so that an exact solution to the modified problem is such fully relaxed ε-efficient decision for the original MOP. For obvious reasons and in order to avoid pathological results, we exclude the case $\varepsilon = 0$ from all the following discussion. Furthermore, we mentioned before that the existence of efficient solutions can, in general, be guaranteed for a closed and bounded set of feasible points in the outcome space, while the discussion of Figure 2 revealed that the set of ε-Pareto outcomes does not necessarily satisfy these conditions. Hence, in order to avoid emptiness of the set of efficient solutions for the problem to be formulated, we choose the set of weakly ε-efficient decisions as the underlying feasible set.

Let the MOP and $\varepsilon \in \mathbb{R}^m$, $\varepsilon \geq 0$ be given. The associated ε-MOP is defined as

$$\varepsilon\text{-MOP: maximize } f(x) \text{ subject to } x \in X_w^{(\varepsilon)}.$$

Analogously to the definition of efficient decisions for the minimization MOP, a feasible decision $x^\circ \in X_w^{(\varepsilon)}$ is called an efficient or weakly efficient solution for the ε-MOP if and only if there does not exist another feasible $x \in X_w^{(\varepsilon)}$ such that $f(x^\circ) \leq f(x)$ or $f(x^\circ) < f(x)$, respectively. The sets of all efficient or weakly efficient solutions for the ε-MOP are denoted by ε-X° and ε-X_w°, respectively, and analogously we define ε-Y° and ε-Y_w° as the corresponding Pareto and weak Pareto sets.

Note that the set of weakly ε-efficient decisions serving as the feasible decision set for the ε-MOP is in general not known, and clearly, without knowledge of this set, solving the ε-MOP is in general not possible. In another paper, Engau and Wiecek (2005a) study the problem of maximizing over the set of ε-efficient solutions in a more abstract setting and propose an alternative characterization of the efficient solution set for the ε-MOP. To clarify terminology, we first define ε-solutions as fully relaxed ε-Pareto outcomes in the following sense.

Let the MOP and $\varepsilon \in \mathbb{R}^m$, $\varepsilon \geq 0$ be given. Then an outcome $y^* \in Y$ is called an ε-outcome if there exists a weak Pareto outcome $y^\circ \in Y_w^\circ$ such that $y^* = y^\circ + \varepsilon$, and the set of all ε-outcomes is denoted by Y^*. A decision $x^* \in X$ is called an ε-decision if it produces an ε-outcome, $y^* = f(x^*) \in Y^*$, and the set of all ε-decisions is denoted by X^*. Formally, we write

$$Y^* = Y \cap (Y_w^\circ + \varepsilon) \text{ and } X^* = \{x \in X : f(x) \in Y^*\}$$

We illustrate the new solutions concepts and sets for the generic outcome set given in Figure 2. Recall that the set of Pareto outcomes Y° is given by the curve connecting points A and B, while all weakly Pareto outcomes in the set Y_w° lie on the curve from C to D. Adding the given vector $\varepsilon \in \mathbb{R}^2$ then yields the set Y^* of ε-outcomes as the curve connecting points G and H. As before, the set $Y_w^{(\varepsilon)}$ of weakly ε-Pareto outcomes is the shaded area enclosed by all marked points and now forms the closed set of attainable outcomes for the ε-MOP. Then the Pareto set for the ε-MOP is given by the curve from E to F without the two endpoints E and F, but including the two points I and J, and finally the set of weak Pareto outcomes ε-Y_w° consists of all points along the complete curve connecting I and J.

Moreover, Figure 2 now suggests that the set of weakly efficient solutions ε-X_w° for the ε-MOP contains the set of those weakly ε-efficient decisions for the MOP that are not ε-efficient and, moreover, that this set includes the set of ε-decisions for the MOP as a (possibly proper) subset. This intuition is confirmed as special case of the results in the aforementioned paper.

Lemma 3.1. *Let the* MOP *and* $\varepsilon \in \mathbb{R}^m$*,* $\varepsilon \geq 0$ *be given. Then all* ε*-decisions and* ε*-outcomes for the MOP belong to the sets of weakly efficient decisions and weak Pareto outcomes for the associated* ε*-MOP,*

$$X^* \subseteq X_w^{(\varepsilon)} \setminus X^{(\varepsilon)} \subseteq \varepsilon\text{-}X_w^\circ \text{ and } Y^* \subseteq Y_w^{(\varepsilon)} \setminus Y^{(\varepsilon)} \subseteq \varepsilon\text{-}Y_w^\circ.$$

Motivated by this lemma, for the remaining discussion we continue to focus on weak efficiency as the underlying solution concept and, in order to solve the ε-MOP, now pursue the development of methods for finding ε-decisions and ε-outcomes for the original MOP. A steady assumption in each of the following four sections is that it is always possible to generate one or more initial (weakly) efficient decisions $x^\circ \in X_w^\circ$ producing a (weak) Pareto outcome $y^\circ = f(x^\circ) \in Y_w^\circ$ for which $y^* = y^\circ + \varepsilon \in Y^*$ is an ε-outcome in the set Y of attainable outcomes. Then, by Lemma 3.1, this outcome $y^* \in Y^*$ is also (weak) Pareto for the ε-MOP, $y^* \in \varepsilon\text{-}Y_w^\circ$.

We emphasize that the main difficulty is not to find an ε-outcome y^* in the set of attainable outcomes, but to identify an associated ε-decision $x^* \in X^*$ that produces this ε-outcome $y^* = f(x^*) \in Y^* \subseteq \varepsilon\text{-}Y_w^\circ$. Now the following four approaches are proposed as possible solution methods.

3.1 Constrained objective approach

Recall the formulation of the MOP and the ε-MOP

$$\text{MOP: minimize } f(x) \text{ subject to } x \in X \subseteq \mathbb{R}^n;$$
$$\varepsilon\text{-MOP: maximize } f(x) \text{ subject to } x \in X_w^{(\varepsilon)}.$$

Now let $y^\circ \in Y_w^\circ$ be a (weak) Pareto outcome for which the associated ε-outcome $y^* = y^\circ + \varepsilon \in Y^*$ is attainable and consider the multiple objective problem

$$\text{CO}(y^*)\text{: maximize } f(x) \text{ subject to } f(x) \leqq y^*, x \in X.$$

Proposition 3.2. *All efficient solutions for the* $\text{CO}(y^*)$ *are* ε*-decisions for the original* MOP*.*

Proof. Observe that $\text{CO}(y^*)$ has the point y^* as a unique Pareto outcome, which by assumption is an attainable ε-outcome, $y^* = y^\circ + \varepsilon \in Y^*$. Thus all efficient decisions x^* for the CO(y^*) must satisfy that $f(x^*) = y^*$, and hence all efficient decisions x^* for the CO(y^*) are ε-decisions for the original MOP, $x^* \in X^*$. $\square$

Hence, Proposition 3.2 now allows to solve the CO(y^*) in order to identify corresponding ε-decisions $x^* \in X^*$ for all ε-outcomes $y^* \in Y^*$, which, by Lemma 3.1, belong to the (weakly) efficient solution set for the ε-MOP, $x^* \in \varepsilon\text{-}X_w^\circ$ and $y^* \in \varepsilon\text{-}Y_w^\circ$.

Note that the above problem formulation resembles the constrained objective scalarization

$$\text{minimize } f_1(x) \text{ subject to } f_i(x) \leq y_i^*, i = 2, \ldots, m, x \in X$$

for which all but one objectives are bounded from above to guarantee that the objective function values improve or at least satisfy certain aspiration levels. However, in the formulation of the CO(y^*) all original objectives remain in the multiple objective function while the objective function levels implied by y^* take the role of reservation levels which must not be exceeded. Finally, observe that once the problem is formulated, it can be solved using any appropriate scalarization method.

3.2 Reference point approach

Perhaps the main difficulty for finding ε-decisions is that the associated ε-outcomes, in general, are interior points of the outcome set, so that many optimization procedures that depend on the fact that optimal solutions occur at the boundary cannot be used. Therefore the above constrained objective approach restricts the set of attainable outcomes in such a way that an ε-outcome can be found at the boundary of the modified outcome set. As a second possibility we suggest to make use of the reference point methodology in multiple objective programming for which, very recently, Lin (2005) investigates the consequences of the location of the chosen reference point with respect to the efficiency of the solutions generated. We base our approach on the weighted ℓ_p norm

$$\ell_p(y, r, w) = \left(\sum_{i=1}^{m} w_i |y_i - r_i|^p \right)^{1/p}$$

which uses a reference point $r \in \mathbb{R}^m$ and a weighting parameter $w \in \mathbb{R}^m$, $w \geq 0$. Here $r \in \mathbb{R}^m$ is usually chosen to be the ideal or a utopia point, and then the absolute values are usually dropped since $r \leqq y$ for all $y \in Y$. In particular for $p = \infty$, the weighted ℓ_∞ norm is defined to be the weighted-Tchebycheff norm introduced in Section 2.2.

Now let $y^\circ \in Y_w^\circ$ be a (weak) Pareto outcome for which the associated ε-outcome $y^* = y^\circ + \varepsilon \in Y^*$ is attainable and choose positive weights $w > 0$ and $r = y^*$ as reference point for the single objective reference point problem

$$\mathrm{RP}(y^*, w)\text{: minimize } \left(\sum_{i=1}^{m} w_i |f_i(x) - y_i^*|^p \right)^{1/p} \text{ subject to } x \in X.$$

Proposition 3.3. *All optimal solutions for the* $\mathrm{RP}(y^*, w)$ *are* ε*-decisions for the original MOP.*

Proof. Observe that since $w > 0$, $\mathrm{RP}(y^*, w)$ has the point y^* as a unique optimal solution, which by assumption is an attainable ε-outcome, $y^* = y^\circ + \varepsilon \in Y^*$. Thus all optimal solutions x^* for the $\mathrm{RP}(y^*, w)$ must satisfy that $f(x^*) = y^*$, and hence all optimal solutions x^* for the $\mathrm{RP}(y^*, w)$ are ε-decisions for the original MOP, $x^* \in X^*$. $\square$

Hence, Proposition 3.3 now allows to solve the $\mathrm{RP}(y^*, w)$ in order to identify corresponding ε-decisions $x^* \in X^*$ for all ε-outcomes $y^* \in Y^*$, which, by Lemma 3.1, belong to the (weakly) efficient solution set for the ε-MOP, $x^* \in \varepsilon\text{-}X_w^\circ$ and $y^* \in \varepsilon\text{-}Y_w^\circ$.

3.3 Projection approach

As mentioned earlier, most traditional optimization methods fail to identify points in the interior of the outcome set and thus need to be modified to allow for the generation of ε-decisions or ε-outcomes which, in general, do not occur at the boundary. In this section we investigate a preliminary projection approach which, besides offering theoretical insight, prepares for more elaborate methods in the subsequent and final section. The main idea is to introduce an additional objective $f_{m+1} : \mathbb{R}^n \to \mathbb{R}$ such that interior points of the original outcome set $Y \subseteq \mathbb{R}^m$ become boundary points of an augmented outcome set $\mathcal{Y} \subseteq \mathbb{R}^{m+1}$. The idea is illustrated by the following example.

Example 3.4. Let the MOP with objective function $f = (f_1, \ldots, f_m)^T$ be given, and introduce an additional component by setting $f_{m+1}(x) = 0$ for all $x \in X$. Then

$$\mathcal{Y} = Y \times \{0\} = f(X) \times \{0\} \subseteq \mathbb{R}^{m+1}$$

is contained in a hyperplane in $m + 1$ dimensional space, and hence all outcomes $y \in Y$ become boundary points $(y, 0)^T \in \mathcal{Y}$ in the augmented outcome space $\mathbb{R}^{m+1}$.

Now let $y^\circ \in Y_w^\circ$ be a (weak) Pareto outcome for which the associated ε-outcome $y^* = y^\circ + \varepsilon \in Y^*$ is attainable and consider the single objective problem

$$\mathrm{PP}(y^*, \rho)\text{: minimize } \mu \text{ subject to } \begin{pmatrix} y^* \\ \rho \end{pmatrix} + \mu \begin{pmatrix} 0 \\ -1 \end{pmatrix} \in \mathcal{Y} = f(X) \times \{0\}$$

with arbitrary $\rho \in \mathbb{R}$. Observe that solving this problem is equivalent to finding the projection of the point $(y^*, \rho)^T \in \mathbb{R}^m \times \mathbb{R}$ onto the set $\mathcal{Y}$ and has $\mu = \rho$ as solution with associated outcome $(y^*, 0) \in \mathcal{Y}$.

Proposition 3.5. *All optimal solutions for the* PP(y^*, ρ) *yield ε-decisions for the original* MOP*.*

Proof. Observe that $\mu = \rho$ is the unique feasible and thus optimal value for the PP(y^*, ρ). The associated solution is $(y^*, 0)^T \in f(X) \times \{0\}$ and thus will be produced by an associated ε-decision x^* for which $f(x^*) = y^*$. □

While clearly trivial from an analytical point of view, Proposition 3.5 now allows to solve the PP(y^*, ρ) numerically in order to identify corresponding ε-decisions $x \in X^*$ for all ε-outcomes $y^* \in Y^*$, which, by Lemma 3.1, belong to the (weakly) efficient solution set for the ε-MOP, $x^* \in \varepsilon\text{-}X_w^\circ$ and $y^* \in \varepsilon\text{-}Y_w^\circ$.

The practical applicability of the above problem is not immediate, however, since the optimal solution is in particular the only feasibly solution. Nevertheless, the idea of augmenting the outcome space and simultaneously lifting the set of attainable outcomes seems quite appealing and thus motivates the following discussion of a general set lifting approach as one further method for the generation of ε-efficient solutions.

3.4 Set lifting approach

We continue to investigate the augmentation idea described in the previous section and propose the formulation of an augmented MOP (AMOP). Then we study the AMOP with respect to two scalarization methods, namely the weighted-sum and the weighted-Tchebycheff norm scalarization, and show how ε-decisions for the MOP can be identified as efficient decisions for the respective AMOPs.

Let the MOP be given, and let $a : Y \subseteq \mathbb{R}^m \to \mathbb{R}$ be a decreasing scalarization function,

$$a(y^1) \geq a(y^2) \text{ whenever } y^1, y^2 \in Y, y^1 \leqq y^2.$$

Then the augmented multiobjective program is defined as

$$\text{AMOP: minimize } \left[f(x), a(f(x))\right]^T \text{ subject to } x \in X,$$

and the function a is called the augmentation function of the AMOP.

The next lemma shows how increasing scalarization functions can be used to define suitable augmentation functions for the AMOP and prepares the two subsequent main theorems of this section.

Lemma 3.6. *Let $s : Y \subseteq \mathbb{R}^m \to \mathbb{R}$ be an increasing scalarization function, and let $y^\circ \in Y$ be an optimal solution for the associated* SOP*. Define*

$$a(y) := (s(y) - s(y^\circ))^{-1}$$

and allow the function value $a(y^\circ) = \infty$. Then the function $a : Y \subseteq \mathbb{R}^m \to \mathbb{R} \cup \{\infty\}$ defines a decreasing augmentation function.

Proof. Let $y^1, y^2 \in Y$ with $y^1 \leqq y^2$ be given. Since s is an increasing scalarization function, this implies $s(y^1) \leq s(y^2)$ and thus $0 \leq s(y^1) - s(y^\circ) \leq s(y^2) - s(y^\circ)$ as y° is an optimal solution for the SOP. Taking inverses gives $(s(y^1) - s(y^\circ))^{-1} \geq (s(y^2) - s(y^\circ))^{-1}$ and thus implies $a(y^1) \geq a(y^2)$. □

The two concluding theorems show that by solving the AMOP with the augmentation function defined from either the weighted-sum or the weighted-Tchebycheff norm scalarization, ε-decisions for the MOP can be obtained as optimal solutions to the scalarized AMOP. Compare the illustration of the weighted-sum set lifting in Figure 3 for further motivation of this approach.

Theorem 3.7. *Let the* MOP *and $\varepsilon \in \mathbb{R}^m$, $\varepsilon \geq 0$ be given, and let $x^\circ \in X_w^\circ$ be a (weakly) efficient decision and $y^\circ = f(x^\circ) \in Y_w^\circ$ be a (weak) Pareto outcome for the* MOP *obtained as optimal solutions for the weighted-sum scalarization* WS(w) *with weighting parameter $w \in \mathbb{R}^m$, $w \geq 0$. Denote*

$$s(f(x), w) := \sum_{i=1}^{m} w_i f_i(x)$$

and let $a(y) := (s(y, w) - s(y^\circ, w))^{-1}$ be the augmentation function for the AMOP*. Let*

$$\epsilon := s(\varepsilon, w) = \sum_{i=1}^{m} w_i \varepsilon_i.$$

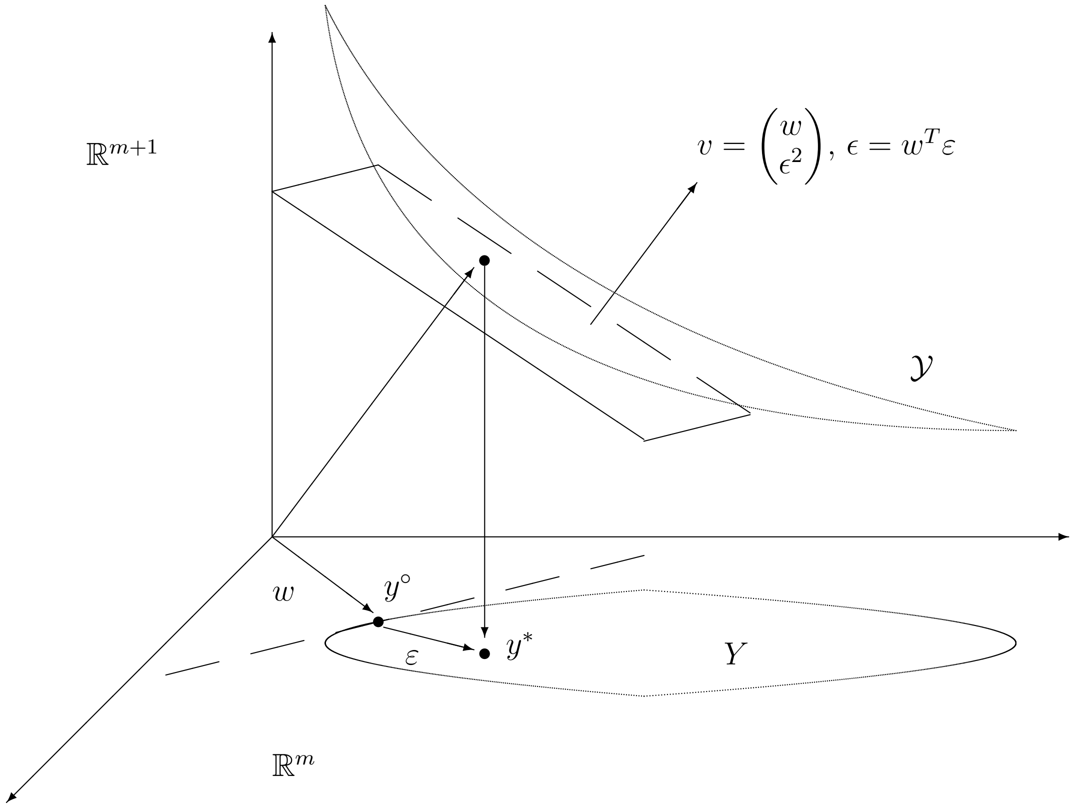

Figure 3: Weighted-sum set lifting method

If the ε-outcome $y^* = y^\circ + \varepsilon \in Y^*$ is attainable for some feasible $x^* \in X$, $y^* = f(x^*)$, then the ε-decision $x^* \in X^*$ for the original MOP *is optimal for the weighted-sum scalarization* $\mathrm{WS}\Big([w, \epsilon^2]^T\Big)$ *of the associated* AMOP.

Proof. Let $y^* = y^\circ + \varepsilon \in Y^*$ be attainable for the original MOP and let $x^* \in X^*$ with $f(x^*) = y^*$ be an associated ε-decision. Now consider the AMOP and the related weighted-sum scalarization

$$\text{AMOP: } \min_{x \in X} \left[f(x), (s(f(x), w) - s(f(x^\circ), w))^{-1}\right]^T$$
$$\mathrm{WS}\Big(\left[w, \epsilon^2\right]^T\Big)\text{: } \min_{x \in X} s(f(x), w) + \epsilon^2 \left(s(f(x), w) - s(f(x^\circ), w)\right)^{-1},$$

and substitute $t = s(f(x), w)$ and $t^\circ = s(f(x^\circ), w)$. Then the objective function of the $\mathrm{WS}\Big([w, \epsilon^2]^T\Big)$ becomes $t + \epsilon^2 (t - t^\circ)^{-1}$, which takes its minimum for $t^* = t^\circ + \epsilon = s(f(x^\circ), w) +$

ϵ, or

$$t^* = s(y^\circ, w) + \epsilon = \sum_{i=1}^{m} w_i y_i^\circ + \sum_{i=1}^{m} w_i \varepsilon_i$$
$$= \sum_{i=1}^{m} w_i (y_i^\circ + \varepsilon_i) = s(y^\circ + \varepsilon, w) = s(y^*, w).$$

Hence we obtain that $y^* \in Y^*$ and thus $x^* \in X^*$ are optimal solutions for the $\mathrm{WS}\Big([w, \epsilon^2]^T\Big)$.
□

Other than the previous result using the weighted-sum scalarization, the final theorem is based on the weighted-Tchebycheff norm and formulated for any (weakly) efficient decision $x^\circ \in X_w^\circ$ or (weak) Pareto outcome $y^\circ \in Y_w^\circ$.

Theorem 3.8. *Let the* MOP *and* $\varepsilon \in \mathbb{R}^m, \varepsilon \geq 0$ *be given, and let* $x^\circ \in X_w^\circ$ *be any (weakly) efficient decision and* $y^\circ = f(x^\circ) \in Y_w^\circ$ *be the associated (weak) Pareto outcome. Denote*

$$s(y, r, w) := \max_{i=1,\ldots,m} \{w_i (y_i - r_i)\}$$

and let $a(y) := (s(y, r, w) - s(y^\circ, r, w))^{-1}$ *be the augmentation function for the associated* AMOP*. Choose* $r = y^\circ$ *as the reference point and set*

$$w_i = \varepsilon_i^{-1} \text{ for } i = 1, \ldots, m$$

with $w_i = \infty$ *if* $\varepsilon_i = 0$*. If the* ε*-outcome* $y^* = y^\circ + \varepsilon \in Y^*$ *is attainable for some feasible* ε*-decision* $x^* \in X^*$ *for the original* MOP *with* $y^* = f(x^*)$*, then* x^* *is optimal for the weighted-Tchebycheff norm scalarization* $\mathrm{TN}\big([y^\circ, 0]^T, [w, 1]^T\big)$ *of the* AMOP*.*

Proof. Let $y^* = y^\circ + \varepsilon \in Y^*$ be attainable for the MOP and $x^* \in X^*$ with $f(x^*) = y^*$ be a corresponding ε-decision. Since the reference point for the augmentation function is chosen as $r = y^\circ$, we first obtain that

$$a(y) = (s(y, y^\circ, w) - s(y^\circ, y^\circ, w))^{-1} = s(y, y^\circ, w)^{-1},$$

and hence the AMOP and the related weighted-Tchebycheff norm scalarization are given by

$$\begin{aligned}\text{AMOP:}\ & \min_{x\in X}\ \left[f(x), s\left(f(x), y^\circ, w\right)^{-1}\right]^T \\ \text{TN}\big([y^\circ,0]^T,[w,1]^T\big)\text{:}\ & \min_{x\in X}\ \max_{i=1,\dots,m} \left\{w_i(f_i(x)-y_i^\circ), s\left(f(x), y^\circ, w\right)^{-1}\right\}\end{aligned}$$

Now observe that the later objective function can equivalently be written as

$$\max\left\{\max_{i=1,\dots,m}\{w_i(f_i(x)-y_i^\circ)\}, s\left(f(x), y^\circ, w\right)^{-1}\right\}$$

and substitute $t = s(f(x), y^\circ, w) = \max_{i=1,\dots,m}\{w_i(f_i(x)-y_i^\circ)\} \geq 0$. Then the objective function of the $\text{TN}\big([y^\circ,0]^T,[w,1]^T\big)$ becomes $\max\{t, t^{-1}\}$ with $t \geq 0$, which takes its minimum for $t = 1$, or

$$t = s(y, y^\circ, w) = \max_{i=1,\dots,m}\{w_i(y_i - y_i^\circ)\} = 1.$$

Note that in particular for $y^* = y^\circ + \varepsilon$, we obtain

$$s(y^*, y^\circ, w) = \max_{i=1,\dots,m}\{w_i(y_i^* - y_i^\circ)\} = \max_{i=1,\dots,m}\{w_i\varepsilon_i\} = 1$$

by our choice of $w_i = \varepsilon_i^{-1}$, and hence, $y^* = y^\circ + \varepsilon \in Y^*$ and thus $x^* \in X^*$ are optimal solutions for the $\text{TN}([y^\circ,0]^T,[w,1]^T)$. □

This concludes the development of proposed exact methods for finding ε-efficient decisions. In order to examine their practical applicability, the following section discusses several results obtained from implementing algorithms for the constrained objective, the reference point and the set lifting approach.

4 Numerical examples

We demonstrate the above methods for the engineering problem of designing a four bar plane truss as given in Figure 4, taken from the treatment in Koski (1988), and compare the results to those obtained from applying the approximate procedure previously proposed in Engau and Wiecek (2005b). While the problem implementation for that paper adopted the formulation given in Coello Coello (2001) who refers back to Stadler and Dauer (1992) but introduces a slight (and probably unintentional) deviation from the original formulation, we corrected the previous implementation for all following results accordingly.

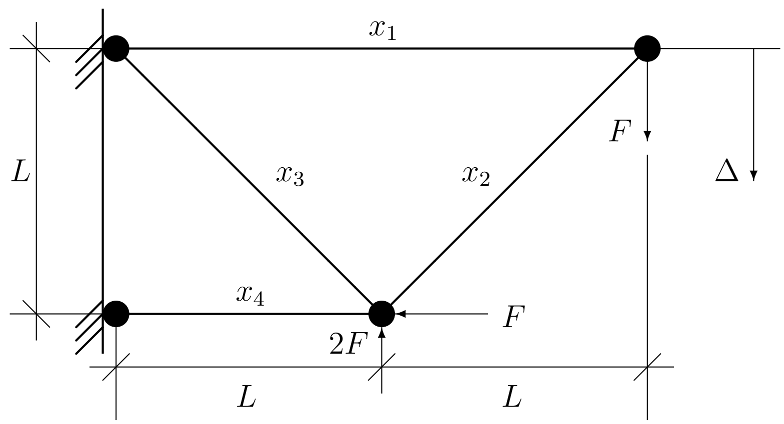

Figure 4: A four bar plane truss.

The problem is formulated as a biobjective program with the two conflicting objectives of minimizing both the volume V of the truss (f_1) and the displacement Δ of the joint (f_2), subject to given physical restrictions on the feasible cross sectional areas x_1, x_2, x_3 and x_4 of the four bars. The stress on the truss is caused by three loading forces of magnitudes F and $2F$ as depicted in Figure 4. The length L of each bar, the Young's modulus of elasticity E and the only nonzero stress component σ are modeled as constants. The mathematical formulation of this problem is given as

$$\begin{aligned}
\text{minimize} \quad & \Big[f_1(x) = L(2x_1 + \sqrt{2}x_2 + \sqrt{2}x_3 + x_4), \\
& f_2(x) = \frac{FL}{E}\left(\frac{2}{x_1} + \frac{2\sqrt{2}}{x_2} - \frac{2\sqrt{2}}{x_3} + \frac{2}{x_4}\right)\Big] \\
\text{subject to} \quad & (F/\sigma) \leq x_1 \leq 3(F/\sigma), \\
& \sqrt{2}(F/\sigma) \leq x_2 \leq 3(F/\sigma), \\
& \sqrt{2}(F/\sigma) \leq x_3 \leq 3(F/\sigma), \\
& (F/\sigma) \leq x_4 \leq 3(F/\sigma).
\end{aligned}$$

with constant parameters $F = 10$ kN, $E = 2{\times}10^5$ kN/cm^2, $L = 200$ cm and $\sigma = 10$ kN/cm^2.

The sets of attainable outcomes for this problem and ε-Pareto outcomes as identified by the approximate weighted-sum method are now depicted in Figure 5, where a detailed description regarding the computation of the latter is provided in the original paper.

Observe that a reasonable relaxation from Pareto to ε-Pareto outcomes does only occur in the middle region of the Pareto curve and that, in general, the number of distinct generated

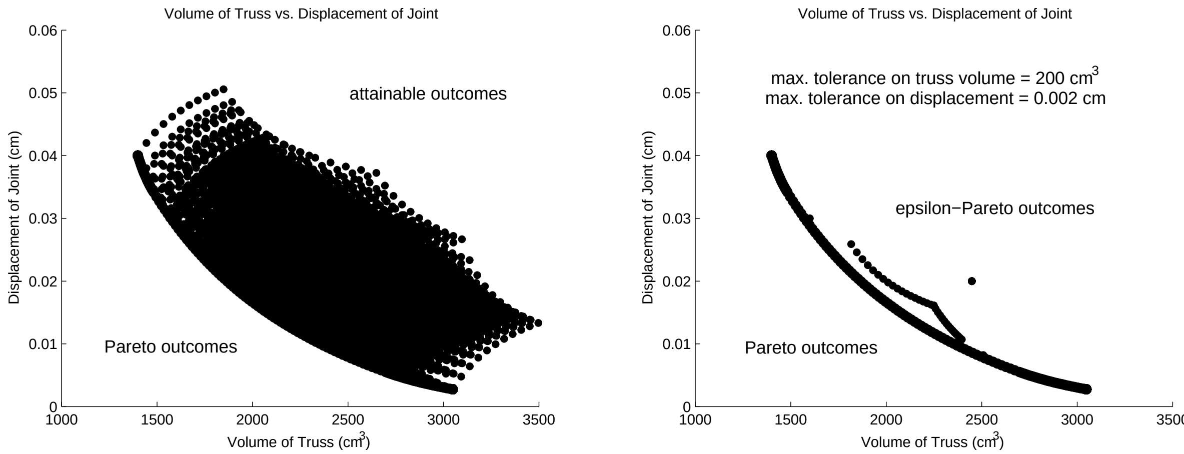

Figure 5: Sets of attainable, Pareto and ε-Pareto outcomes (approximate method)

outcomes is much greater for the Pareto than for the ε-Pareto outcomes. Moreover, many Pareto outcomes are also identified when solving the problem with specified tolerances, and it turns out that a further increase of the specified tolerances for truss volume and joint displacement yields an even further reduction in the number of identified ε-Pareto outcomes. As mentioned before, we believe that parts of these methodological issues might result from the difficulty of deciding on ϵ-optimality when solving the SOP due to insufficient knowledge of accurate lower bounds on the optimal solution. Consequently, although the obtained set does include several points that relax efficiency in the desired manner, in general the proposed approximate methods do not provide a satisfying representation of the set of ε-Pareto outcomes and thus reinforce the necessity for exact methods as developed in this paper.

Figures 6 and 7 show the results obtained when applying the constrained objective and the reference point method, respectively, for the four different tolerances $\varepsilon^1 = (200, 0.002)^T$, $\varepsilon^2 = (500, 0.005)^T$, $\varepsilon^3 = (800, 0.008)^T$ and $\varepsilon^4 = (1000, 0.01)^T$. As intended, both approaches yield a reasonable set of fully relaxed ε-outcomes for each of these four different values.

Recall that the formulation of the constrained objective approach guarantees that the identified solutions in decision or outcome set are truly ε-efficient or ε-Pareto, respectively, while targeting at a fully relaxed ε-efficient solution. The reference point method, on the other hand, merely attempts to get as close as possible to some ε-Pareto outcome, thereby possibly exceeding those ε-constraints imposed by the constrained objective approach. This observation explains the different behavior of these two methods in the lower right regions of Figures 6 and 7.

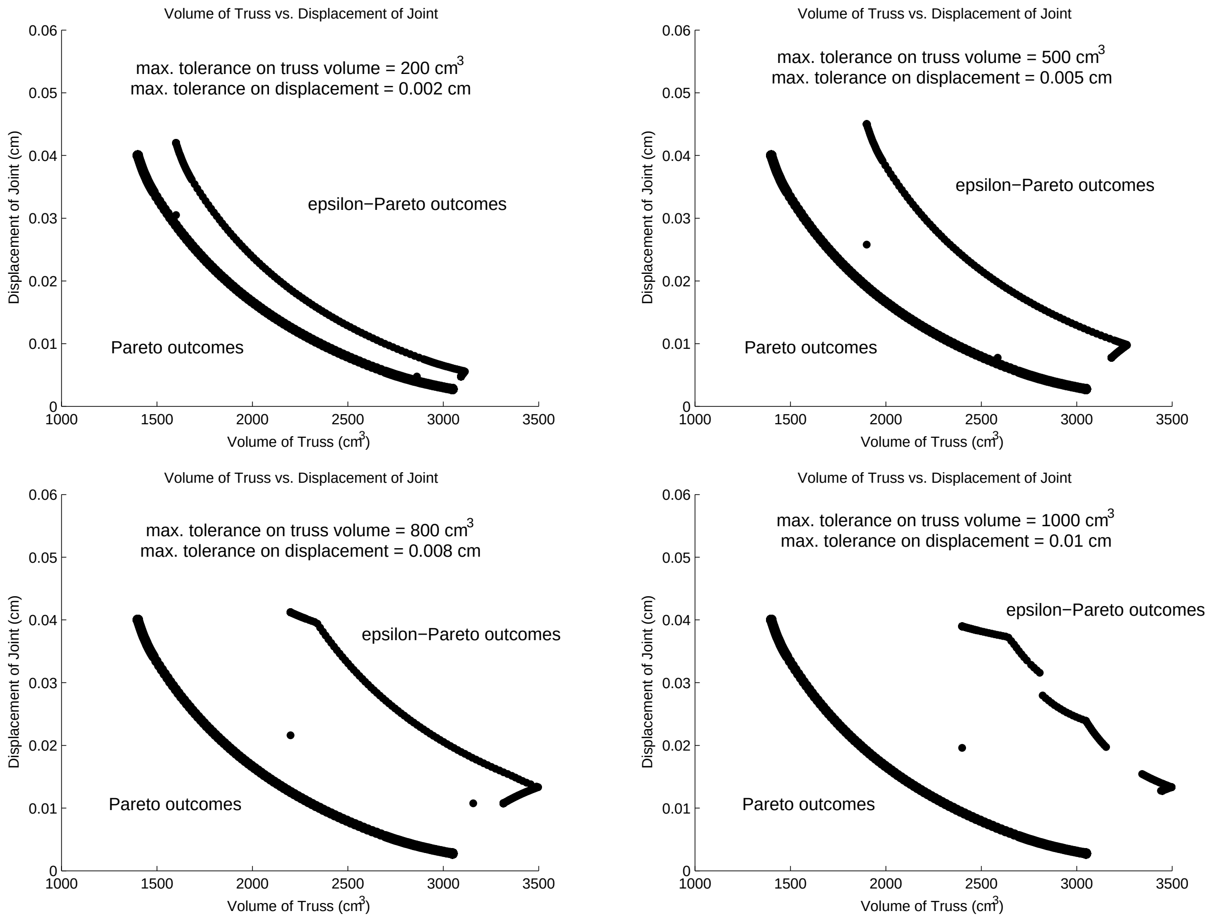

Figure 6: Results using the constrained objective approach (exact method)

Another difference between the constrained objective and the reference point approach is that the first remains a multiple objective program and thus requires a previous scalarization before subsequent optimization. Based upon the scalarization function used, this might lead to additional numerical difficulties possibly explaining the few outliers not occurring when applying the reference point method. In general, however, both approaches resolve the existing drawbacks of the approximate methods and yield an almost complete representation of the set of ε-outcomes or fully relaxed ε-Pareto outcomes for various choices of the tolerances ε over the whole attainable set.

The results shown in Figure 8 are obtained from implementing the weighted-sum set lifting approach and reveals that this method, although conceptually appealing, is subject to various numerical issues and difficulties. In particular, by potentially allowing a division by zero, this formulation becomes very sensitive and hence does not seem suitable for actual

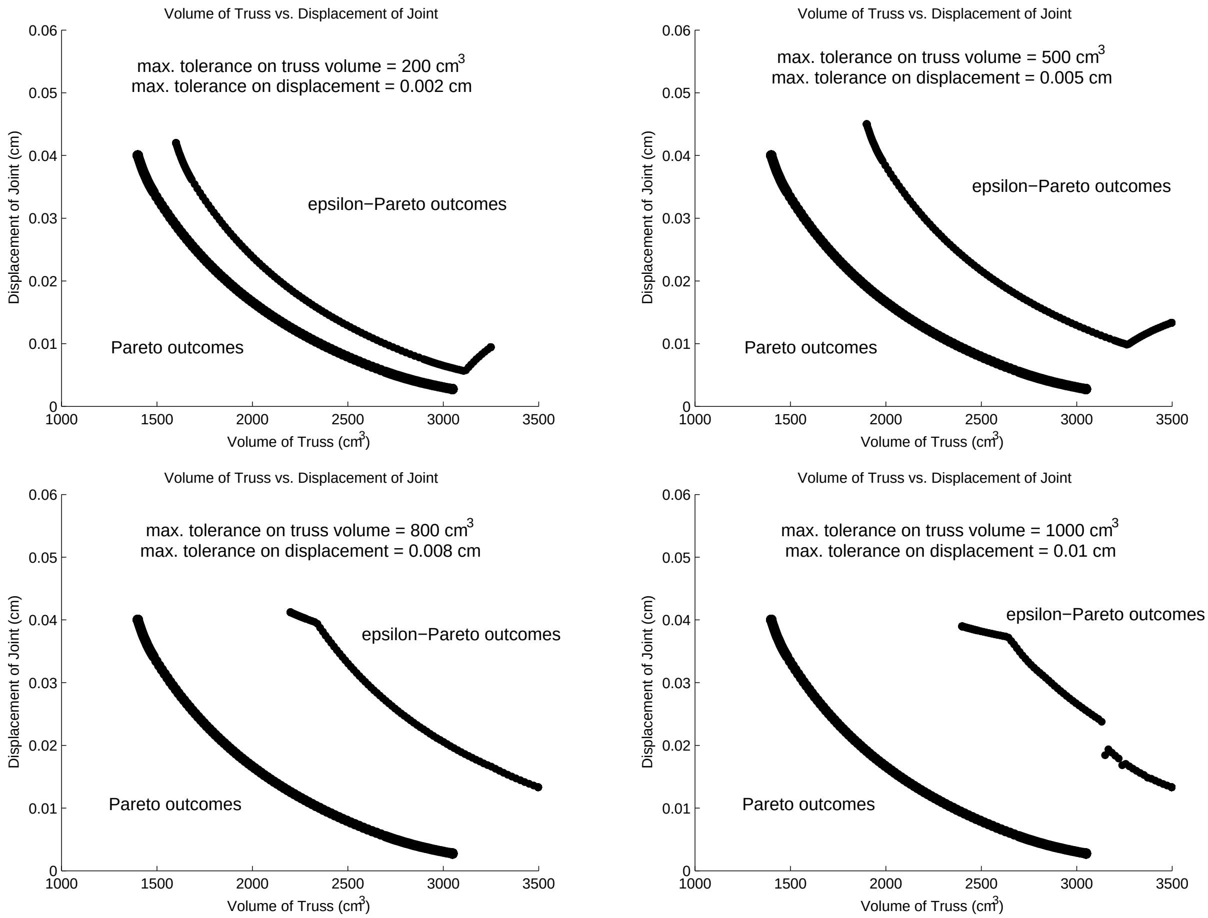

Figure 7: Results using the reference point approach (exact method)

implementation.

5 Conclusions

In many real world situations, multiobjective programs can only be solved in approximation. Since the set of efficient decisions commonly consists of infinitely many points, only an approximate representation can be found in form of a finite set of points. Furthermore, these points themselves may only be approximate to efficient decisions due to computational restrictions, and thus the importance of good approximation methods in multiobjective programming is well recognized, see Ruzika and Wiecek (2005). Nevertheless, searching the literature suggests that applicable methods to actually identify approximate solutions have not been well developed.

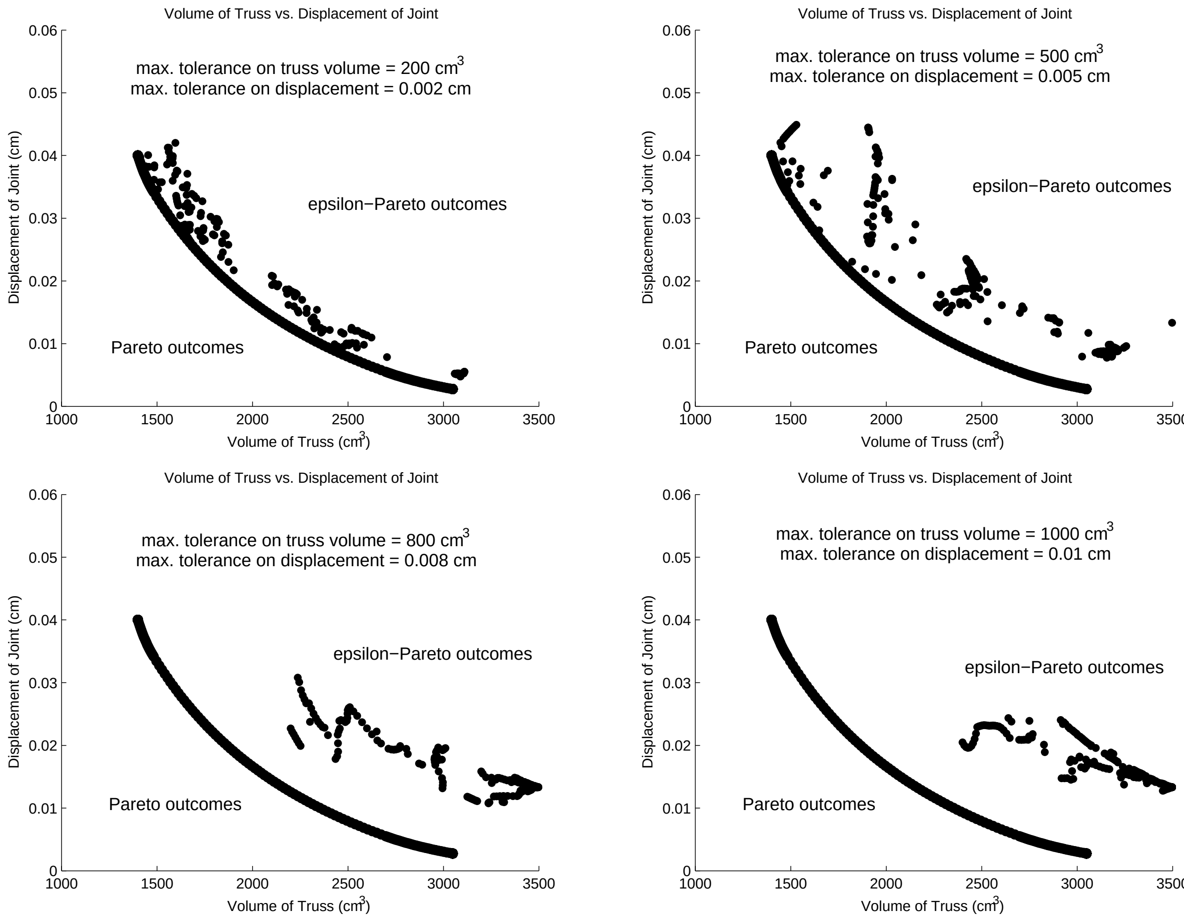

Figure 8: Results using the set lifting approach (exact method)

Following preliminary work presented in Engau and Wiecek (2005b) which also emphasizes the relevance of suboptimal solutions for the assessment of sensitivity and practical decision making situations dealing with collections of more than just one multiple objective program, this paper continues the development of generating methods for ε-efficient solutions and gives theoretical proofs for their capability of finding particularly those solutions achieving the full relaxation specified by the tolerance vector ε. Application of these methods to an engineering design problem shows that various drawbacks of those method derived earlier are resolved and therefore leads to the conclusion that the new methods are also practically suited for that purpose.

We believe that the formulation of other methods for generating ε-efficient solutions is only limited by personal creativity and thus encourage interested researchers to propose new approaches towards this goal. Our personal future interest lies in further application of the concept of ε-efficiency for different decision making situations and new solution methodologies for multiple objective programming problems.

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